Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024

SCOPUS

Role of Environmental Contamination in Measles Transmission dynamics

Avik Bag[1], Jayanta Mondal[2], and Amar Nath Chatterjee[13]

[1]Department of Mathematics, Raja Rammohun Roy Mahavidyalaya, Burdwan University, West Bengal-712406, India.
[2]Department of Mathematics, Diamond Harbour Women's University, Sarisha, WestBengal-743368, India.
[3]Department of Mathematics, K. L. S. College, Nawada, Magadh University, Bodh Gaya, Bihar-805110, India.
*Corresponding Author: A. N. Chatterjee (anchaterji@gmail.com)

Abstract

Measles disease is one of the biggest communicable diseases and is still responsible for 2.598 million deaths every year. In this regard, our research focuses on measles disease transmission dynamics and the impact of indirect contact rates on the environment. In this paper, we propose a nonlinear mathematical model for measles disease and analyze a deterministic epidemiological measles model for control of the disease using vaccination. Here we executed the equilibrium points of the model, and we got two equilibrium points, namely, a disease-free equilibrium point (DEFP) and a unique endemic equilibrium point (EEP), and we also analyzed the local stability of DFEP and EEP by center manifold theory. Here we have also shown the global stability of DFEP and EEP by theCastillo-Chavez criterion and Lasalle invariant principle, respectively. In this study, we executed the basic reproduction number. if the basic reproduction number is less than unity otherwise the system shows a significant outbreak. Numerical illustrations demonstrate that if the rate of environmental contamination increased, then the number of infected people also increased. But if the environment is disinfected by sanitization then the number of infected people cannot drastically increase.

1 Introduction

Since the day of creation, humans have been in danger due to different types of epidemic outbreaks. Measles, a highly contagious respiratory illness caused by the rubeola virus, continues to threaten global public health despite the availability of a safe and effective vaccine. The disease is easily spread through the air when an infected person sneezes, coughs, and talks, and can remain in the air for up to two hours [1]. It is a virus of paramyxovirus family, genus morbilivirus, which is found only in the human body among all animal species. Measles is spread easily from person to person, and an individual can contract the virus by breathing in air contaminated with the virus or by touching a surface contaminated with the virus and then touching their mouth, nose or eyes [2]. The symptoms of an individual caused by a virus are such as fever and cough runny nose and red and watery eyes that usually appear that usually develop 10-12 days after exposure to an infectious person. In severe cases, it can lead to complications such as pneumonia, encephalitis, and deafness. This disease is more dangerous for children under five years of age and adults older than 20 years of age. There is no specific treatment for measles inspite of availability of a safe and

cost-effective vaccine Today, measles is a common and fatal disease in the world. It spreads person to person at a high rate of over 90% among vulnerable people, such as young children, pregnant women, and those with weak immune systems due to the mode of transmission and infectious properties. Some vaccines, including two doses of MMR (measles, mumps, and rubella), are nearly 100% effective at preventing measles and will protect 99% of people from measles. This means that a person must get both doses in order to receive the full benefits of the vaccine. But once a person recovers, they will not be able to get measles again. The measles vaccine is safe, effective, and expensive.

Mathematical modelling has played an important role in understanding how to predict the spread of measles. In [3], a mathematical model was presented using the real incidence data from pakistan. The study found stability conditions(depending on R_0) proposed a strategy to control spread, calculated the sensitivity of R_0, and suggested ways to improve vaccine efficacy and coverage. The authors of [4] studied a modified SVEIR measle model to control measle outbreak in Bangladesh with double dose vaccination. They found that two equilibria exist: disease-free and endemic, and the latter persists if R_0 remains above one. A mathematical model to examine the impact of preventive measures on the control of measles is studied in [5] and they have shown that the disease free state is globally stable if the reproduction number $R_0 \leq 1$, meaning the measles will die out eventually. However when $R_0 > 1$, the disease will spread. Bai and Liu [6] studied a discrete time measles model with a periodic transmission rate by using the basic reproduction number as the threshold for disease persistence. Xue et al. [7] used the model to investigate the periodic outbreak in mainland china, they recommended enhancing vaccination and implementing an optimal control system to minimize infections. The authors of [8] analyzed the seasonal spread of measles in china using a mathematical model, that was used to analyze the dynamics of the disease depending on R_0, and to simulate the monthly data of reported cases of measles in china. [9] carried out a study on predicting and preventing of measles disease epidemics in New Zealand. In their work they used a deterministic SIR model to model the dynamics of measles disease under varying immunization strategies in a population with size and age structure.

In this paper, in section 2, we proposed a deterministic mathematical model. In section 3, we have done positivity and boundedness of the model system.In section 4, executed equilibrium points. In section 5, we have done, existence and uniqueness of solution of the system. Also basic reproduction number of the system is executed in this section. In section 6, local stability of disease free equilibrium point is analyzed along with existence of endemic equilibrium point and local stability of endemic equilibrium point are done. Persistence of the system is also verified in this section. Here we analyzed global stability of disease free equilibrium point.In section 7, we have also shown the global stability of endemic equilibrium point. In addition, in section 8, numerical simulations are done and in the

2 Mathematical Model Formulation

In this section, we proposed a deterministic model clarified by contaminated environment to investigate measles transmission dynamics. The total human individuals at t represented by N(t) is classified into four subpopulation, Susceptible S(t), Vaccinated V(t), Infected I(t), Recovered R(t) i.e N(t)=S(t)+V(t)+I(t)+R(t). To show that measles can be minimized by maintaining the contaminated environment, we incorporate another important subclass of contaminated

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

environment C(t), To better understand, the transmission dynamics of measles virus, some investigation are focused on the contaminated environment, C(t). Daily recruitment into the susceptible class is at a rate Λ. Individual in the susceptible class receive a vaccination at a rate γ and lose immunity at rate α. The transmission rate of susceptible individuals is β, and the force of infection term is βSI. The interaction rate at which susceptible population with contaminated environment is ψ. Natural mortality rate of all subpopulation is δ. τ is the rate at which vaccinated population becomes infected. η is the mortality rate caused by measles. The rate at which infected population become recovered is ω. μ is the mortality rate of infected population caused by measles. η is the rate at which vaccinated individual becomes recovered. κI is the infected individual that makes environment contaminated at rate κ and $\delta_c$is the rate at which contaminated environment become uninfected.

S - Susceptible, V - Vaccinated, I - Symptomatic Infected, R - Recovered, C - environmental contamination δ_c= while the removal of infection from the market is given by

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta SI - \xi CS + \alpha V - (\gamma + \delta)S, \\
\frac{dV}{dt} &= \gamma S - \tau\beta IV - (\alpha + \eta + \delta)V, \\
\frac{dI}{dt} &= \beta SI + \xi CS + \tau\beta IV - (\omega + \mu + \delta)I, \\
\frac{dR}{dt} &= \omega I + \eta V - \delta R, \\
\frac{dC}{dt} &= \kappa I - \delta_c C,
\end{aligned} \qquad (1)$$

with non negative initial value

$$S(t_0) = S_0 > 0, V(t_0) = V_0 \geq 0, I(t_0) = I_0 \geq 0, R(t_0) = R_0 \geq 0, C(t_0) = C_0 \geq 0, \qquad (2)$$

Infectious individuals in the I classes contaminate the environment with measles at the rates κ. The virus is cleared from the contaminated environment at the rate δ_c.

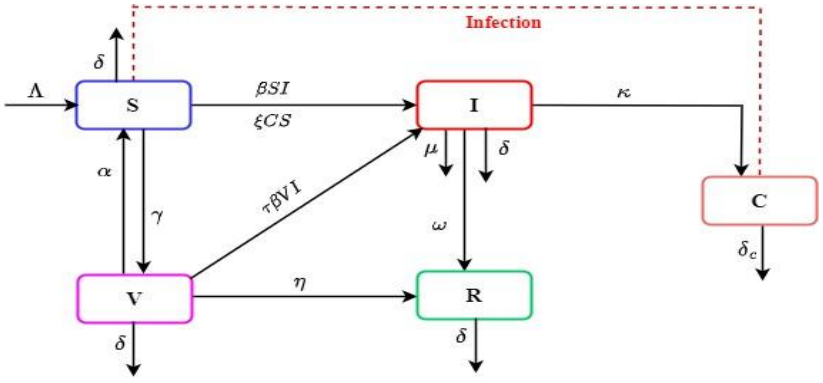


Figure 1: Schematic diagram of the mathematical model for the transmission dynamics of the Measles Infection.

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

3 Mathematical analysis

3.1 Feasibility of the model

Theorem 1. *The system* (1) *is invariant in* $\mathbb{R}^5_+$

Proof. From our system (1) we observe that

$$\begin{aligned} \frac{dS}{dt}|_{S=0} &= \Lambda + \alpha V \geq 0, \quad \frac{dV}{dt}|_{V=0} = \gamma S \geq 0, \quad \frac{dI}{dt}|_{I=0} = \xi CS \geq 0 \\ \frac{dR}{dt}|_{R=0} &= \omega I + \eta V \geq 0, \quad \frac{dC}{dt}|_{C=0} = \kappa I \geq 0 \end{aligned} ,$$

Hence $\mathbb{R}^5_+$ is an invariant set.

□

3.2 Positivity

Theorem 2. *If the solutions of* (1) *with initial values* (2) *satisfying* $S(t) > 0, V(t) > 0, I(t) > 0, R(t) > 0, C(t) > 0$ *for all* $t > 0$. *Then the system (1) is positively invariant and attracting in* $\mathbb{R}^5_+$.

Proof. First equation of system (1) we can be written as

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \xi CS + \alpha V - (\gamma + \delta)S, \\ &\geq -\beta SI - \xi CS - \gamma S - \delta S, \end{aligned}$$

Thereafter by integration, we obtain the following equation

$$S(t) \geq S(t_0) \exp[-\int_0^t (\beta I + \xi C + \gamma + \delta) ds].$$

Which is positive as $S(t_0) \geq 0$ for $t > 0$.
Further from the second equation of the system (1) we can get

$$\begin{aligned} \frac{dV}{dt} &= \gamma S - \tau \beta IV - \alpha V - \eta V - \delta V, \\ &\geq -V(\tau \beta I + \alpha + \eta + \delta). \end{aligned}$$

Which leads to

$$V(t) \geq V(t_0) \exp[-\int_0^t (\tau \beta I + \alpha + \eta + \delta) ds].$$

Which implies that $V(t_0) \geq 0$ for $t > 0$.
In similar way from third equation of system (1) we get

$$\begin{aligned} \frac{dI}{dt} &= \beta SI + \xi CS + \tau \beta IV - (\omega + \mu + \delta)I, \\ &\geq (\omega + \mu + \delta)I. \end{aligned}$$

So,

$$I(t) \geq I(t_0) \exp[-\int_0^t (\omega + \mu + \delta) ds].$$

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

Therefore $I(t) > 0$ as $I(t_0) \geq 0$ for $t > 0$.
From fourth equation of system (1)

$$\begin{aligned} \frac{dR}{dt} &= \omega I + \eta V - \delta R, \\ &\geq -\delta R \end{aligned}$$

Which leads to $R(t) > 0$ as $R(t_0) \geq 0$ for $t > 0$. Again similarly

$$\begin{aligned} \frac{dC}{dt} &= \kappa I - \delta_c C, \\ &\geq -\delta_c C. \end{aligned}$$

$$C(t) \geq C(t_0) \exp[-\int_0^t (\delta_c) ds].$$

Therefore $C(t) > 0$ as $C(t_0) \geq 0$ for $t > 0$. Thus all the solutions of system (1) remain non-negative for all finite time, that is, for all $t > 0$ Hence proof. □

3.3 Boundedness

In this subsection we study the boundedness of the system (1) as none of the population can not grow unboundedly. To do this we state the theorem that assured that the solutions of system (1) is bounded if we start with non-negative initial conditions.

Theorem 3. *All the solutions of* (1) *with non-negative initial values* (2) *that starts in* $\mathbb{R}^5_+$ *are uniformly bounded in the* Θ*, where* Θ *is defined in the proof.*

Proof. Here we shall show that all the feasible solutions are uniformly bounded in Θ. From the positivity of solutions, it is clear that

$$\frac{dS}{dt} \leq \Lambda - (\gamma + \delta)S.$$

Which implies that

$$\limsup_{t \to \infty} S(t) \leq \frac{\Lambda}{\gamma + \delta}.$$

Adding the first four equation ($N = S + V + I + R$) yields

$$\frac{dN}{dt} \leq \Lambda - \delta N.$$

So

$$\limsup_{t \to \infty} N(t) \leq \frac{\Lambda}{\delta}.$$

Now from last equation,

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

$$\begin{aligned}\frac{dC}{dt} &= \kappa I - \delta_c C. \\ &\leq \frac{\Lambda K}{\delta} - \delta_c C.\end{aligned}$$

This implies that

$$\limsup_{t\to\infty} C(t) \leq \frac{\Lambda K}{\delta\delta_c}.$$

So we get positively invariant set

$$\Theta = \left\{(S,V,I,R,C) \in \mathbb{R}^5_+ : S \leq \frac{\Lambda}{\gamma+\delta},\ S+V+I+R \leq \frac{\Lambda}{\delta},\ C \leq \frac{\Lambda K}{\delta\delta_c}\right\}.$$

Therefore the solutions in $\mathbb{R}^5_+$ will enter and remain in the region Θ for all finite time. Thus the dynamics of system (1) can be considered in Θ.Hence the system is well-posed and biologically realistic. □

4 Equilibrium point

There are two types of equilibria of the system (1), namely

(a) The disease free equilibria $E_0(S_0,V_0,0,R_0,0)$, where

$$\begin{aligned}S_0 &= \frac{\Lambda}{\gamma+\delta} + \frac{\alpha\gamma\Lambda}{(\gamma+\delta)(\alpha\delta+\eta\gamma+\eta\delta+\gamma\delta+\delta^2} \\ V_0 &= \frac{\gamma\Lambda}{\alpha\delta+\eta\gamma+\eta\delta+\gamma\delta+\delta^2} \\ R_0 &= \frac{\eta\gamma\Lambda}{\delta(\alpha\delta+\eta\gamma+\eta\delta+\gamma\delta+\delta^2)}\end{aligned}$$

(b) The interior equilibrium point $E^*= (S^*,V^*,I^*,R^*,C^*)$

4.1 Existence and Uniqueness of solution

The general first-order ODE is in the form:

$$x' = f(t,x), x(t_0) = x_0$$

One will be interested in asking the question that under what condition there exist an unique solution. To answer these, let

$$\begin{aligned}f_1 &= \Lambda - \beta SI - \xi CS + \alpha V - (\gamma+\delta)S, \\ f_2 &= \gamma S - \tau\beta IV - (\alpha+\eta+\delta)V, \\ f_3 &= \beta SI + \xi CS + \tau\beta IV - (\omega+\mu+\delta)I, \\ f_4 &= \omega I + \eta V - \delta R, \\ f_5 &= \kappa I - \delta_c C,\end{aligned} \qquad (3)$$

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

we show that: $\frac{\partial f_i}{\partial x_j}\ i,j = 1,2,3,4,5$ are continuous and bounded i.e the partial derivatives are continuous and bounded.

Therefore

$$\begin{aligned}
\left|\frac{\partial f_1}{\partial S}\right| &= |-\beta I - \xi C - \gamma - \delta| < \infty,\ \left|\frac{\partial f_1}{\partial V}\right| = |\alpha| < \infty,\ \left|\frac{\partial f_1}{\partial I}\right| = |-\beta S| < \infty \\
\left|\frac{\partial f_1}{\partial R}\right| &= |0| < \infty,\ \left|\frac{\partial f_1}{\partial C}\right| = |-\xi S| < \infty,\ \left|\frac{\partial f_2}{\partial S}\right| = |\gamma| < \infty,\ \left|\frac{\partial f_2}{\partial V}\right| = |-\tau\beta I| < \infty, \\
\left|\frac{\partial f_2}{\partial I}\right| &= |-\tau\beta V| < \infty,\ \left|\frac{\partial f_2}{\partial R}\right| = |0| < \infty,\ \left|\frac{\partial f_2}{\partial C}\right| = |0| < \infty,\ \left|\frac{\partial f_3}{\partial S}\right| = |\beta I + \xi C| < \infty \\
\left|\frac{\partial f_3}{\partial V}\right| &= |\tau\beta I| < \infty,\ \left|\frac{\partial f_3}{\partial I}\right| = |\beta S + \tau\beta V - (\omega + \mu + \delta)| < \infty,\ \left|\frac{\partial f_3}{\partial R}\right| = |0| < \infty, \\
\left|\frac{\partial f_3}{\partial C}\right| &= |\xi S| < \infty,\ \left|\frac{\partial f_4}{\partial S}\right| = |0| < \infty,\ \left|\frac{\partial f_4}{\partial V}\right| = |\eta| < \infty,\ \left|\frac{\partial f_4}{\partial I}\right| = |\omega| < \infty, \\
\left|\frac{\partial f_4}{\partial R}\right| &= |-\delta| < \infty,\ \left|\frac{\partial f_4}{\partial C}\right| = |0| < \infty,\ \left|\frac{\partial f_5}{\partial S}\right| = |0| < \infty,\ \left|\frac{\partial f_5}{\partial V}\right| = |0| < \infty, \\
\left|\frac{\partial f_5}{\partial I}\right| &= |0| < \infty,\ \left|\frac{\partial f_5}{\partial R}\right| = |0| < \infty,\ \left|\frac{\partial f_5}{\partial C}\right| = |-\delta_c| < \infty.
\end{aligned},$$

We have established that all these partial derivative are continuous and bounded. Hence we can say that there exist a unique solution.

5 Basic reproductive number of the model

The Basic reproductive number (R_0) is the average number of secondary infections caused by a single infectious individual during their entire infectious lifetime. This number is dimensionless and calculated at the DFE by the next generation matrix method. In the present work, We derived the threshold quantity known as reproductive number which is denoted by (R_0). For this we assemble the compartments which are infected from system (1), and decomposing the right hand side as -, where transmission part, expressing the production of new infection, and the transition part, describe the change in state. It can be represented by the spectral radius of the largest magnitude of the next generation matrix. According to the system in model (1) there are two infectious compartments with (2 × 2) matrices.

$$\begin{aligned}
\frac{dI}{dt} &= \beta SI + \xi CS + \tau\beta IV - (\omega + \mu + \delta)I = f_1 - v_1 \\
\frac{dC}{dt} &= \kappa I - \delta_c C = f_2 - v_2.
\end{aligned}.$$

where

$f_1 = \beta SI + \xi CS + \tau\beta IV$, $v_1 = (\omega + \mu + \delta)I$, $f_2 = 0$, $v_2 = -\kappa I + \delta_c C$.

Our system (1) defined as follows

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

$$F_1 = \begin{pmatrix} \beta SI + \xi CS + \tau\beta IV \\ 0 \end{pmatrix} \text{and} V_1 = \begin{pmatrix} (\omega + \mu + \delta)I \\ -\kappa I + \delta_c C \end{pmatrix}.$$

Now F and V can be written as

$$F = \begin{pmatrix} \beta S + \tau\beta V & \xi S \\ 0 & 0 \end{pmatrix}. \quad V = \begin{pmatrix} \omega + \mu + \delta & 0 \\ -\kappa & -\delta_c \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} \frac{\delta_c(\beta S + \tau\beta V) + \xi k S}{\delta_c(\omega + \mu + \delta)} & \frac{-\xi S}{\delta_c} \\ 0 & 0 \end{pmatrix}.$$

Thus the mathematical definition of Basic reproductive number is the largest eigen value of the next generation matrix FV^{-1}

$$\rho(FV^{-1}) \equiv R = \frac{\delta_c(\beta S + \tau\beta V) + \xi k S}{\delta_c(\omega + \mu + \delta)}$$

$$R_0 = \frac{\delta_c(\beta S_0 + \tau\beta V_0) + \xi k S_0}{\delta_c(\omega + \mu + \delta)} \equiv \frac{\Lambda(\alpha\delta + \eta\gamma + \eta\delta + \gamma\delta + \alpha\gamma + \delta^2)(\xi\kappa + \beta\delta_c) + \tau\beta\gamma\Lambda\delta_c(\gamma + \delta)}{\delta_c(\gamma + \delta)(\alpha\delta + \eta\gamma + \eta\delta + \gamma\delta + \delta^2)(\omega + \mu + \delta)}$$

6 Stability analysis of the model equilibria

6.1 Local stability of disease free equilibrium point

Theorem 4. *The disease free equilibrium point* $E_0(S_0,V_0,0,R_0,0)$ *of system* (1) *is locally asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

Proof. To determine the local stability of E_0, we compute the Jacobian matrix of the system (1) around E_0.

The characteristic equation of Jacobian matrix at E_0 is $det(J_{E0} - \lambda I_5) = 0$.

$(\lambda + \delta)(\lambda + \gamma + \delta)(\lambda + \eta + \delta + \alpha)\{(\beta S_0 + \tau\beta V_0 - \omega - \mu - \delta - \lambda)(-\delta_c - \lambda) - \xi\kappa S_0\} = 0$

From the above expression three eigen values are $-\delta$, $-\gamma - \delta$, $-\alpha - \eta - \delta$ and other two eigen values can be expressed in the for as follows:

$$\frac{\xi\kappa S_0}{(\beta S_0 + \tau\beta V_0 - \omega - \mu - \delta - \lambda)(-\delta_c - \lambda)} = 1$$

By resetting the above equation, we have

$$\begin{aligned} B(\lambda) &= \frac{\xi\kappa S_0}{(\beta S_0 + \tau\beta V_0 - \omega - \mu - \delta - \lambda)(-\delta_c - \lambda)} \\ &= R_1(\lambda) \end{aligned}$$

Let us consider $\lambda = x + iy$with $Re(\lambda) \geq 0$, we have

$$|R_1(\lambda)| \leq \frac{\xi\kappa S_0}{|(\beta S_0 + \tau\beta V_0 - \omega - \mu - \delta - \lambda)||(-\delta_c - \lambda)|} \leq R_1(x) \leq R_1(0)$$

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

From the above expression, we can write that $R_1(0) = B(0) = R_0 < 1$, which leads to the form $|R_1(\lambda)| \leq 1$. Therefore for the basic reproduction number $R_0 < 1$, all the eigen values corresponding to the characteristic equation $B(\lambda) = 1$ are all real and no imaginary roots. Thus for $R_0 < 1$, all eigen values have negative real parts and so the disease-free equilibrium point is locally asymptotically stable. Now we consider the case when basic reproduction number $R_0 > 1$, that is, $B(0) > 1$, which indicates the fact that

$$\lim_{\lambda \to \infty} B(\lambda) = 0.$$

Then there exist atleast one eigen value $\lambda_s > 0$ in such a way that $B(\lambda_s) = 1$. This implies that there exist atleast one nonnegative eigen value $\lambda_s > 0$ for the variational matrix J_{E0}. Thus the disease-free equilibrium point is unstable for
$R_0 > 1$. □

6.2 Existence of Endemic equilibrium point

The endemic equilibrium point (EEP) is evaluated by considering all the state variables must not be zero at the equilibrium state which means EEP $E^* = (S^*, V^*, I^*, R^*, C^*)$ and equating all the equations of the model to be zero. Thus the Endemic Equilibrium Point E^*as follows:

$$S^* = \frac{\Lambda M_1}{M_1 M_2 - \alpha\gamma}, \quad V^* = \frac{\gamma\Lambda}{M_1 M_2 - \alpha\gamma}, \quad R^* = \frac{\omega I^*}{\delta} + \frac{\eta\gamma\Lambda}{\delta(M_1 M_2 - \alpha\gamma)}, \quad C^* = \frac{\kappa I^*}{\delta_c},$$

where $I^*$is the positive root of the equation

$$AI^{*2} + BI^* + D = 0,$$

where

$$A = \tau\beta\left(\frac{\beta\delta_c + \xi\kappa}{\delta_c}\right)$$

$$B = \left(\frac{\beta\delta_c + \xi\kappa}{\delta_c}\right)(\alpha + \eta + \delta) + \tau\beta(\gamma + \delta) - \frac{\tau\beta\Lambda}{\delta_c}\frac{\beta\delta_c + \xi\kappa}{\omega + \mu + \delta}$$

$$D = \{(\alpha + \eta + \delta)(\gamma + \delta) - \alpha\gamma\}\{1 - R_0\}$$

So we get a positive root of I^*if $D < 0$ which means $R_0 > 1$ where

$$S_0 = \frac{\Lambda}{\gamma + \delta}, \quad M_1 = \tau\beta I^* + \alpha + \eta + \delta, \quad M_2 = \beta I^* + \frac{\xi\kappa}{\delta_c} I^* + \gamma + \delta,$$
$$M_1 M_2 - \alpha\gamma > 0, \quad (\alpha + \eta + \delta)(\gamma + \delta) - \alpha\gamma > 0.$$

Theorem 5. *The endemic equilibrium $E^*$of the system* (1) *is locally asymptotically stable if $R_0 > 1$.*

Proof. Introducing $x_1 = S(t), x_2 = V(t), x_3 = I(t), x_4 = R(t), x_5 = C(t)$ the system (1) becomes

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

$$\begin{aligned}
\frac{dx_1}{dt} &= \Lambda - \beta x_1 x_3 - \xi x_5 x_1 + \alpha x_2 - (\gamma + \delta)x_1 \equiv q_1 \\
\frac{dx_2}{dt} &= \gamma x_1 - \tau\beta x_3 x_2 - (\alpha + \eta + \delta)x_2 \equiv q_2 \\
\frac{dx_3}{dt} &= \beta x_1 x_3 + \xi x_5 x_1 + \tau\beta x_3 x_2 - (\omega + \mu + \delta)x_3 \equiv q_3 \\
\frac{dx_4}{dt} &= \omega x_3 + \eta x_2 - \delta x_4 \equiv q_4 \\
\frac{dx_5}{dt} &= \kappa x_3 - \delta_c x_5 \equiv q_5
\end{aligned}$$

with $R_0 = 1$ and choosing the bifurcation parameter β the Jacobian matrix around DFEP E_0 at the threshold point $\beta = \beta^* = \frac{\delta_c(\omega+\mu+\delta)-\xi\kappa S^0}{\delta_c(S^0+\tau V^0)}$ is given by

$$JE0 = \begin{bmatrix} -\gamma-\delta & \alpha & -\beta*x1 & 0 & -\xi x1 \\ \gamma & -\alpha-\eta-\delta & -\tau\beta^* x_2 \beta^* x_1 + & 0 & 0 \\ 0 & \eta & \tau\beta^* x_2 - \omega - \mu - \delta & 0 & \xi x1 \\ 0 & 0 & \omega & -\delta & \\ & & \kappa & 0 & \\ 0 & & & -\delta_c & \end{bmatrix}$$

Now we shall find right eigen vector and left eigen vector corresponding to zero eigen value, The right eigen vector is

$$\begin{aligned}
W &= (W_1, W_2, W_3, W_4, W_5)^T \\
&= \left(\frac{(\alpha+\eta+\delta)(\kappa\delta-\omega\delta_c)}{\eta\gamma} + \frac{\tau\beta x_2\delta_c}{\kappa\gamma}, \frac{\kappa\delta-\omega\delta_c}{\eta}, -\frac{\delta_c}{\kappa}, 1, -1\right)^T
\end{aligned}$$

and the left eigen vector is

$$\begin{aligned}
V &= (V_1, V_2, V_3, V_4, V_5)^T \\
&= \left(1, \frac{\alpha}{\alpha+\eta+\delta}, 0, 0, -\frac{\xi x_1}{\delta_c}\right)
\end{aligned}$$

Hence we have

$$a = \sum_{k,i,j=1}^{5} v_k w_i w_j \frac{\partial^2 q_k}{\partial x_i \partial x_j} E_0, \quad b = \sum_{k,i=1}^{5} v_k w_i \frac{\partial^2 q_k}{\partial x_i \partial \beta} E_0$$

whose sign determined the local stability criteria of the Endemic equilibrium point E^*. Substituting the values of all second order partial derivative measured at DFE is given by $a = \beta w_1 w_3 v_1 - \xi w_1 w_5 v_1 - \tau\beta v_2 w_2 w_3 < 0$ and $b = -w_3 x_1 - w_3 \tau x_2 > 0$ provided $\kappa\delta < \omega\delta_c$ and $\xi < \frac{\delta_c(\omega+\mu+\delta)}{\kappa S^0}$

So a transcritical bifurcation occurs at $R_0 = 1$.

Hence EEP is locally asymptotically stable if $R_0 > 1$. □

7 Persistence

We proved that while basic reproductive number $R_0 < 1$, then the measles disease dies out irrespective of the initial size of the epidemic. If $R_0 > 1$, the disease-free equilibrium E_0 become

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

unstable. Usually it is considered that the infected individuals $I(t)$,$C(t)$ will remain persistent for this event. Now, we prove the following theorem to verify the persistence of the measles disease.

Theorem 6. *For* $R_0 > 1$, *the infection will be uniformly persistence which means that there exist an* $\rho > 0$ *in such away that for non-negative solutions of* (1), *satisfies the following* $\liminf I(t) > \rho$, $\liminf C(t) > \rho$. *Furthermore, there exist an interior steady state in this case.*

Proof. To prove the disease persistence we apply the theorem by H.R.Thieme. In order to prove this, we consider that

$G = (S,V,I,R,C), G^{-} = (I,C),$

$B = \{G \in \mathbb{R}^5_+ : G_i \geq 0, i = 1,2,3,4,5,$ where G_i is the i-th component of $G\}$

$B_0 = \{G \in B : G_i > 0, i = 3,5\}$

$D = B/B_0 = \{G \in B : G_i = 0 \text{ for } i = 3,5\}$

Now we want to show that the system (1) is uniformly persistent with respect to (B_0,D). Since D contain an unique equilibrium E_0, it is sufficient to show that $W^s(E_0)^T B_0 = \emptyset$ where $W^s(E_0)$ denotes the stable manifold of the disease free equilibrium E_0.

Suppose this is not true. Then there is a solution $(S(t),V(t),I(t),R(t),C(t)) \in B_0$ the system (1), such that

$$\lim_{t\to\infty} (S(t),V(t),I(t),R(t),C(t)) \to (S_0,V_0,0,R_0,0)$$

From the system (1) we have,

$$\begin{pmatrix} dI/dt \\ dC/dt \end{pmatrix} = \begin{pmatrix} \beta SI + \tau\beta IV + \xi CS \\ 0 \end{pmatrix} + \begin{pmatrix} -(\omega+\mu+\delta) & 0 \\ \kappa & -\delta_c \end{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix}$$

$$\geq \begin{pmatrix} \beta S_0 + \tau\beta V_0 - (\omega+\mu+\delta) & \xi S_0 \\ \kappa & -\delta_c \end{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix} \equiv J_{E_1}\bar{G}$$

So,

$$J_{E_0} = \begin{pmatrix} \beta S_0 + \tau\beta V_0 - (\omega+\mu+\delta) & \xi S_0 \\ \kappa & -\delta_c \end{pmatrix}.$$

Note that J_{E0} is equal to $F - V$, has atleast one eigen value with positive real part when $R_0 > 1$. So there exist solutions for the linear system $\frac{d\bar{G}}{dt} = J_{E_0}\bar{G}$ that can proliferate exponentially. Due to comparison argument solution of G never be bounded for $t \to \infty$ which provides a contradiction to the reality that the solutions of (1) are uniformly bounded. Thus we have $W^s(E_0)^T B_0 = \emptyset$. Thus we can conclude that the model (1) is uniformly persistent. □

8 Sensitivity analysis

Sensitivity analysis is a tool to determine how different values of an independent variable affect a particular dependent variable under a given set of assumptions. Sensitivity analysis is defined as

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

follows: Let U be a variable that depends on a parameter p, then the normalized sensitivity index of variable U with respect to the parameter p is given by the following equation:

$$\Gamma_p^U = \frac{\partial U}{\partial p} \times \frac{p}{R} \tag{4}$$

Now from our model the Basic reproductive number is as follows:

$$R_0 = \frac{\delta_c(\beta S_0 + \tau\beta V_0) + \xi k S_0}{\delta_c(\omega + \mu + \delta)} \tag{5}$$

and the local sensitivity analysis of R_0 with respect to each parameter is calculated as follows:

Sensitivity index of β is as follows:

$$\Gamma_\beta^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} \tag{6}$$

$$= \frac{\delta_c(\beta S_0 + \tau\beta V_0)}{\delta_c(\beta S_0 + \tau\beta V_0) + \xi\kappa S_0} > 0 \tag{7}$$

Similarly sensitivity index of $\delta_c$is as follows:

$$\Gamma_{\delta_c}^{R_0} = \frac{\partial R_0}{\partial \delta_c} \times \frac{\delta_c}{R_0}$$

$$= -\frac{\xi\kappa S_0}{\delta_c(\beta S_0 + \tau\beta V_0) + \xi\kappa S_0} < 0$$

Similarly we can show that the sensitivity index of $\xi,\kappa,\tau>0$ and sensitivity index of $\delta,\mu,\omega,\delta_c<0$. Also it is to be observed that R_0 is independent of the system parameter η, that is sensitivity index of $\eta = 0$.

So, we can conclude that β,ξ,τ,κhave a positive sensitivity indices. Thus they have a great effect on the transmission dynamics and prevalence of measles. Here increment of β,ξ,κ,τwill increase the value of the basic reproduction number R_0. The increase of $\delta,\mu,\omega,\delta_c$will cause the decrease of R_0.Therefore they have a high influence on controlling and preventing transmission dynamics.

9 Global stability analysis

9.1 Global stability of DFE point

To justify the DFE point is globally asymptotically stable, we apply the castillochavez criterion. Based on the theorem, to investigate the global stability of DFEP we rewrite the model in equation (1) as follows:

$$\frac{dX}{dt} = F(X,\ Z)$$
$$\frac{dZ}{dt} = G(X,\ Z), G(X,\ 0) = 0,$$

where $X = (S,\ V,\ R) \in \mathrm{R}^3$ represent the number of uninfected population while $Z = (I,\ C) \in \mathrm{R}^2$ represent the number of infected population. For the disease free equilibrium point to be globally stable it should meet the following two axiom:

$A_1.\frac{dX}{dt} = F(X^*, 0)$ where $X^*$is globally asymptotically stable.

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

$$A_2.\frac{dZ}{dt} = \frac{\partial G}{\partial Z}(X^*,\ 0)Z - \widehat{G}(X,\ Z), \widehat{G}(X,\ Z) \geq 0 \forall (X,\ Z) \in \Omega$$

Theorem 7. *The equilibrium point* $E_1 = (S_1,V_1,0,R_1,0)$ *of the system in equation* (1) *is globally asymptotically stable if it satisfies the three conditions of Castillo-Chavez criterion.*

Proof. To prove the equilibrium point $E_1 = (S_1,V_1,0,R_1,0)$ is globally asymptotically stable, first of all we should identify $F(X, Z)$ and $G(X, Z)$. From our model

$$F(X, Z) = \Lambda + \alpha V - (\gamma + \delta)S - \delta R$$
$$\gamma S - (\alpha + \eta + \delta)V \eta V - \delta R$$
$$G(X, Z) = \beta SI + \xi CS + \tau \beta IV - (\omega + \mu + \delta)I$$
$$\kappa I - \delta_c C$$

The first condition is already proved. Now

$$X^* = \left(\frac{\Lambda}{\gamma + \delta} + \frac{\alpha\gamma\Lambda}{(\gamma + \delta)(\alpha\delta + \eta\gamma + \eta\delta + \gamma\delta + \delta^2)}, \frac{\gamma\Lambda}{\alpha\delta + \eta\gamma + \eta\delta + \gamma\delta + \delta^2}, \frac{\eta\gamma\Lambda}{\delta(\alpha\delta + \eta\gamma + \eta\delta + \gamma\delta + \delta^2)}\right),$$

is satisfying the second condition of Castillo-Chavez criterion for the reduced system $\frac{dX}{dt} = F(X^*, 0)$. From the first equation in system (1), we have the following equation:

$$\frac{dS}{dt} = \Lambda + \alpha V - (\gamma + \delta)S.$$

Integrating both sides

$$S = \frac{\exp^{-(\gamma+\delta)(t+c)}}{-(\gamma + \delta)} + \frac{\Lambda + \alpha V}{\gamma + \delta}. \quad (8)$$

Since as $t \to \infty$

$$\exp^{-(\gamma+\delta)(t+c)} \to 0 and S \to \frac{\Lambda + \alpha V}{\gamma + \delta} = S_1. \quad (9)$$

From the second equation in system (1), we have the following equation:

$$\frac{dV}{dt} = \gamma S - (\alpha + \eta + \delta)V. \quad (10)$$

Integrating both sides using the method of separable variable,

$$V = \frac{\exp^{-(\alpha+\eta+\delta)(t+c)}}{-(\alpha + \eta + \delta)} + \frac{\gamma S}{(\alpha + \eta + \delta)} \quad (11)$$

Since $t \to \infty$,

$$\exp^{-(\alpha+\eta+\delta)(t+c)} \to 0 and V \to \frac{\gamma S}{(\alpha + \eta + \delta)} = V_1.$$

From the third equation, we have,

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

$$\frac{dR}{dt} = \eta V - \delta R. \tag{12}$$

Integrating both sides using the method of separation of variables,

$$R = \frac{\exp^{-\delta(t+c)}}{-\delta} + \frac{\eta V}{\delta}. \tag{13}$$

Since $t \longrightarrow \infty$,

$$\exp^{-\delta(t+c)} \to 0 and R \to \frac{\eta V}{\delta} = R_1.$$

Using the same fashion we can show that $I \longrightarrow 0 = I_1$, ,$C \longrightarrow 0 = C_1$.

Thus the second condition of Castillo-Chavez criterion holds true. Lastly we must show the third criterion as follows: i.e

$\frac{\partial G}{\partial Z}(X^*, 0)$ is an M-matrix and $G_b(X,Z) \geq 0 \; \forall \; (X,Z) \in \Omega$.

Now

$$\frac{\partial G}{\partial Z}(X^*, 0) = \begin{pmatrix} \frac{\partial G_1}{\partial I} & \frac{\partial G_1}{\partial C} \\ \frac{\partial G_2}{\partial I} & \frac{\partial G_2}{\partial C} \end{pmatrix}. \tag{14}$$

So

$$\frac{\partial G}{\partial Z}(X^*, \; 0) = \begin{pmatrix} \beta S + \tau \beta V - (\omega + \mu + \delta) & \xi S \\ \kappa & -\delta_c \end{pmatrix}. \tag{15}$$

Since the non-diagonal entries of $\frac{\partial G}{\partial Z}(X^*,0)$ are non-negative. Thus $\frac{\partial G}{\partial Z}(X^*,0)$ is an M-matrix.

From

$$\frac{dZ}{dt} = \frac{\partial G}{\partial Z}(X^*, 0) - \widehat{G}(X, Z).$$

We have obtained the following equation:

$$\widehat{G}(X, Z) = \begin{pmatrix} \beta(S_1 - S) + \tau\beta(V_1 - V) & \xi(S_1 - S) \\ 0 & 0 \end{pmatrix}. \tag{16}$$

Since, $S_1 \geq S$ and $V_1 \geq V$ it is clear that $G_b(X, Z) \geq 0$ for all $(X, Z) \in \Omega$.
Therefore the DFEP is globally asymptotically stable. □

9.2 Global stability o Endemic Equilibrium point

Theorem 8. *The Endemic Equilibrium point is globally asymptotically stable.*

Proof. To prove global stability of EE point we use Lyapunov function as follows:

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

$$\begin{aligned}\varphi(E^*) &= \left(S - ln\frac{S}{S^*}\right) + \left(V - ln\frac{V}{V^*}\right) + \left(I - ln\frac{I}{I^*}\right) + \left(R - ln\frac{R}{R^*}\right) + \left(C - ln\frac{C}{C^*}\right)\\ \frac{d\varphi}{dt} &= \frac{d\varphi}{dS}\frac{dS}{dt} + \frac{d\varphi}{dV}\frac{dV}{dt} + \frac{d\varphi}{dI}\frac{dI}{dt} + \frac{d\varphi}{dR}\frac{dR}{dt} + \frac{d\varphi}{dC}\frac{dC}{dt}\\ &= \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{V^*}{V}\right)\frac{dV}{dt} + \left(1 - \frac{I^*}{I}\right)\frac{dI}{dt} + \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt} + \left(1 - \frac{C^*}{C}\right)\frac{dC}{dt}.\end{aligned}$$

After simplification and calculation we get

$$\frac{d\varphi}{dt} = U - V,$$

where

$$\begin{aligned}U &= \Lambda + \kappa I + \beta S^* I + \xi C S^* + \gamma S^* + \delta S^* + \tau\beta I V^*\\ &\quad + \alpha V^* + \eta V^* + \delta V^* + \omega I^* + \mu I^* + \delta I^* + \delta R^* + \delta_c C^*\\ V &= \delta S + \delta V + \mu I + \delta I + \delta R + \delta_c C + \frac{S^*}{S}\Lambda + \frac{S^*}{S}\alpha V\\ &\quad + \frac{V^*}{V}\gamma S + \beta S I^* + \tau\beta V I^* + \frac{I^*}{I}\xi C S + \frac{R^*}{R}\omega I + \frac{R^*}{R}\eta V + \frac{C^*}{C}\kappa I.\end{aligned}$$

Hence if

$$U - V \leq 0, \text{ then } \frac{d\varphi}{dt} \leq 0.$$

Whenever $S = S^*$, $V = V^*$, $I = I^*$, $R = R^*$, $C = C^*$

$$\frac{d\varphi}{dt} = 0.$$

Thus by Lasalle invariant principle $E^*$is globally asymptotically stable if $U - V \leq 0$ □

10 Numerical simulation and Discussion

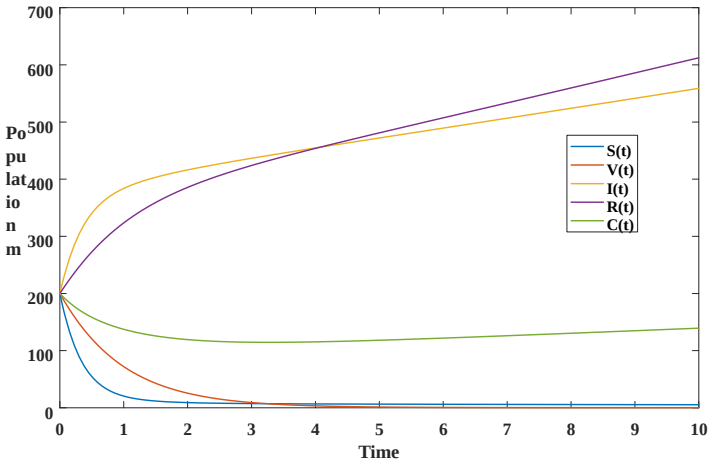


Figure 2: Graphical representation of total population when R_0 <1

Corrosion Management ISSN: 1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

In this section, numerical simulations are done by studying and interpreting the effects of some of the biological parameters by considering different values. In figure 2, we have plotted the graph of all sub-population when basic reproduction number $R0$ <1, where we observe that infected and recovered population increases but infected population become stable after a short period of time but recovered population increases. In fig 3, we have shown the graphical representation of all sub-population when basic reproduction number $R0$ >1. In this case infected population increases rapidly. Recovered and contaminated environment also increases but rate of increase is lower than infected population and after some time they become stable., but susceptible and vaccinated population goes to extinction. In figure 4 and figure 5 we have plotted graphically dynamics of susceptible and vaccinated population for different values of β. Here we observe that wheβ increases then susceptible and vaccinated population decreases. In figure 6, we have plotted a graph that is showing when β increases then infected population increases and after going to its peak, start decreasing and become stable. In figure 7, we observe that, rate of increase of infected population is higher with contaminated environment than without Table 1: The parameter values utilized in the numerical simulations for system
(1)

Parameters	Biological definition	value (million/year)	Source
Λ	Rate of recruitment of the susceptible population	20	Estimated
β	Disease transmission rate	0.004	Estimated
ξ	Interaction rate of Susceptible population with contaminated environment	0.01	Estimated
α	Vaccine waning rate	0.16	Estimated
δ	Natural death rate	0.0875	Estimated
γ	Mortality rate of Susceptible population due to measles	0.06	Estimated
τ	Rate at which vaccinated population becomes infected	0.003	Estimated
η	Rate at which vaccinated population becomes recovered	0.8	Estimated
ω	Rate at which infected population becomes recovered after getting vaccine	0.14286	Estimated
μ	Mortality rate of infected population due to measles	0.125	Estimated
κ	Rate at which infected population makes environment contaminated	0.21	Estimated
δ_c	Rate at which contaminated environment becomes uninfected	0.81	Estimated

Corrosion Management ISSN:1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

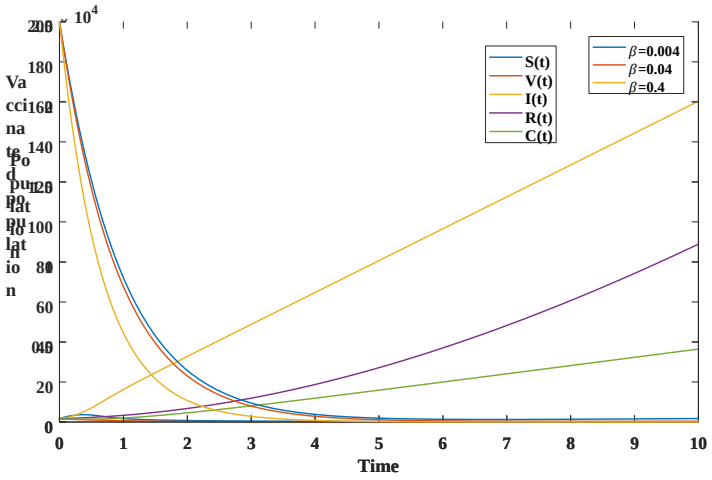


Figure 3: Graphical representation of total population when $R_0 > 1$

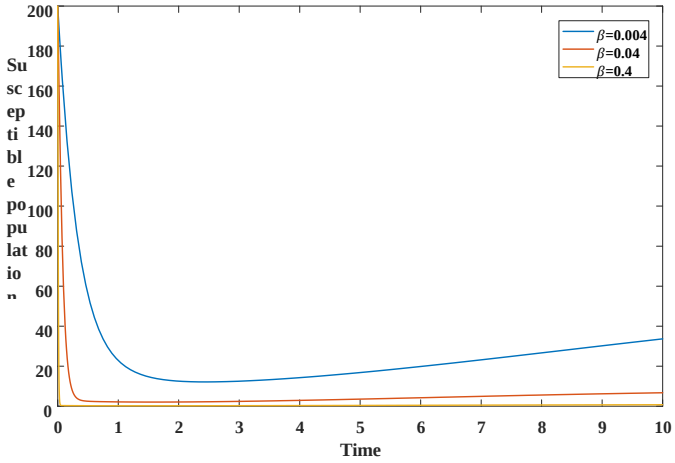
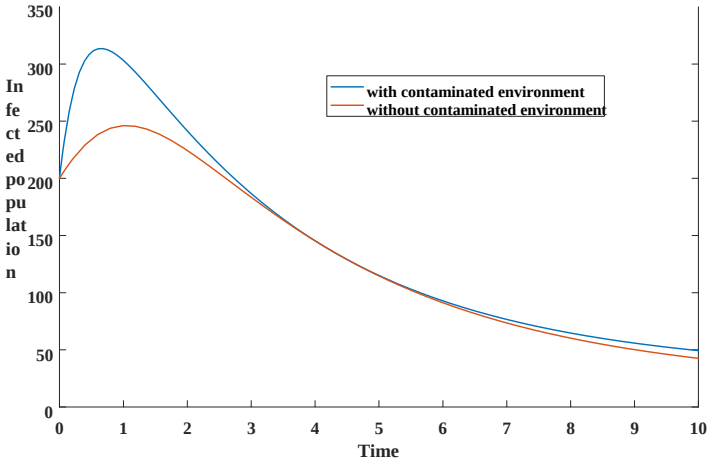


Figure 4: Effect of varying β on Susceptible population

contaminated environment.. In figure 8 and figure 9, we have seen the effect of varying ξ on infected and susceptible population. when ξ decreases, the rate of infected population decreases and the rate of susceptible population increase.

In this article, we present a mathematical model to study the transmission dynamics of measles. The human population is divided into four compartments: Susceptible, vaccinated, infected and recovered, and another compartment is taken as contaminated environment. This

work analyzedmathematical model to study the transmission and recovery dynamics of measles



infection using Figure 5: Effect of varying β on Vaccinated population

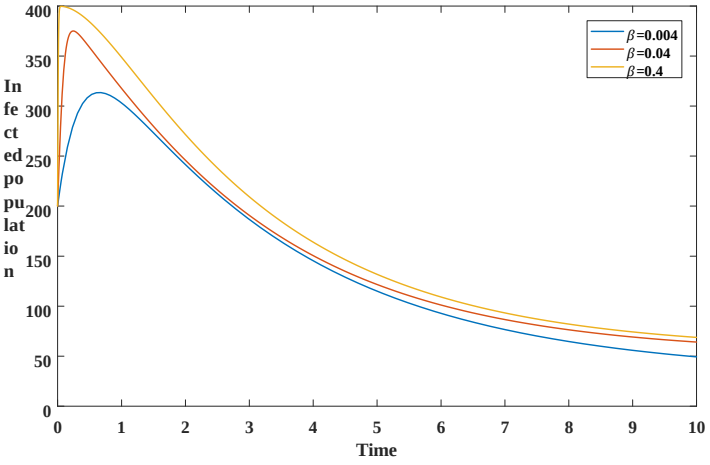


Figure 6: Effect of varying β on Infected population

ordinary differential equations(ODEs) to discuss its boundedness and stability. In this paper we have established the existence of non-negative solutions of the mathematical model. The basic reproduction number, R_0 is calculated and used to determine the stability of the disease free equilibrium and the persistence of the disease.it is shown that the disease free equilibrium point will be locally asymptotically stable when the basic reproduction number, R_0 <1, otherwise unstable. The model has unique endemic equilibrium which is locally asymptotically stable if R_0

Corrosion Management ISSN:1355-5243
(https://corrosion-management.com/)
Volume 34, Issue 01 – June 2024
SCOPUS

>1.We also have shown the local stability Figure 7: Infected population with and without contamination of environment

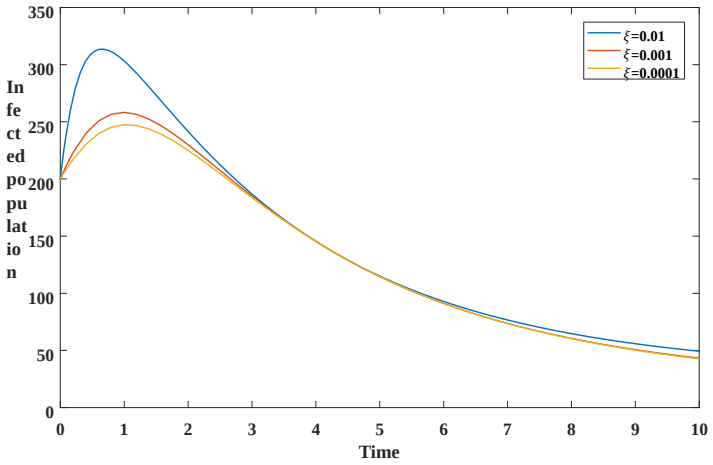


Figure 8: Effect of varying ξ on infected population

of DFEP and EEP by center manifold theory and global stability by castillochavezcriterion.In addition, numerical simulation of the model was presented and discussed.

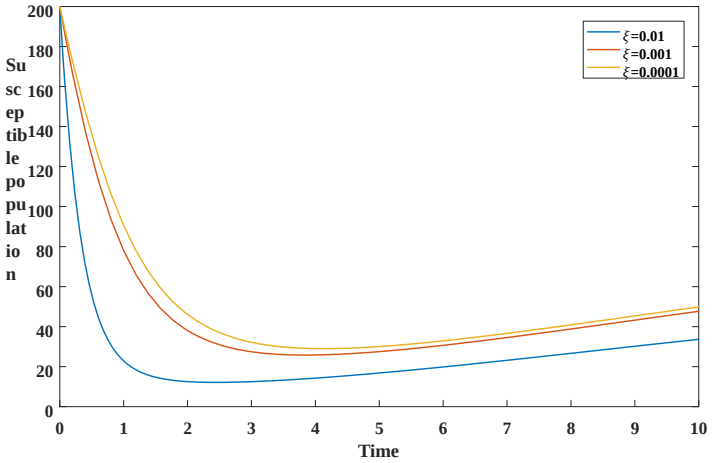


Figure 9: Effect of varying ξ on susceptible population

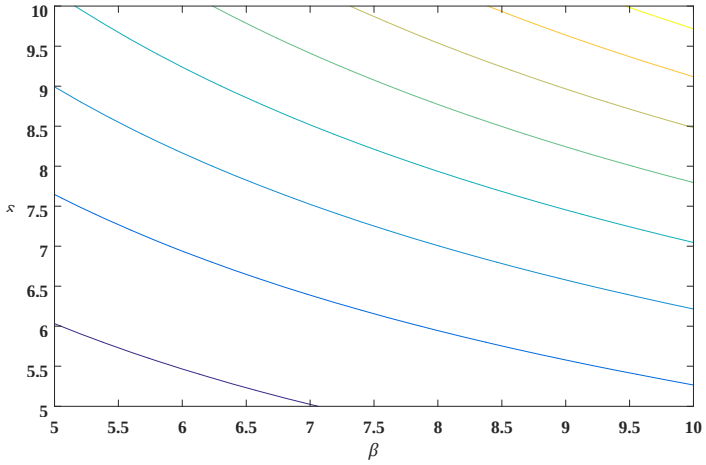


Figure 10: Contour plot of basic reproductive number $R0$ as a function of β and δ

References

[1] M. A. Ibrahim and A. D´enes, "Stability and threshold dynamics in a seasonal mathematical model for measles outbreaks with double-dose vaccination," *Mathematics*, vol. 11, no. 8, p. 1791, 2023.

[2] D. D. Flannery, A. Z. Barboza, K. C. Wade, M. R. Pfeifer, J. S. Gerber, J. S. Morris, and K. M. Puopolo, "Measles serostatus among parturient patients

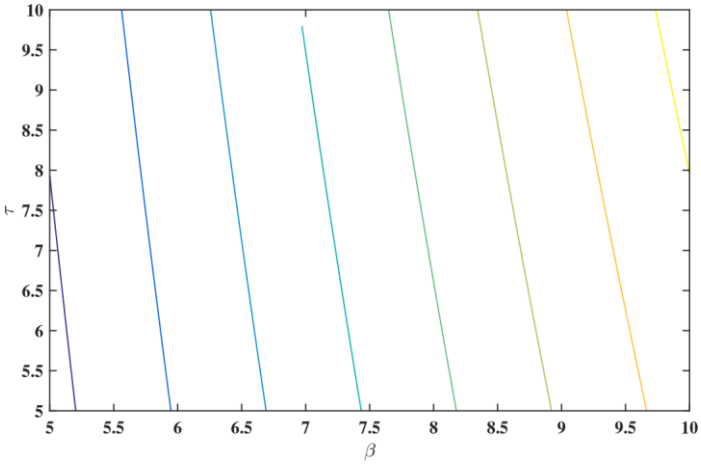


Figure 11: Contour plot of basic reproductive number $R0$ as a function of β and τ

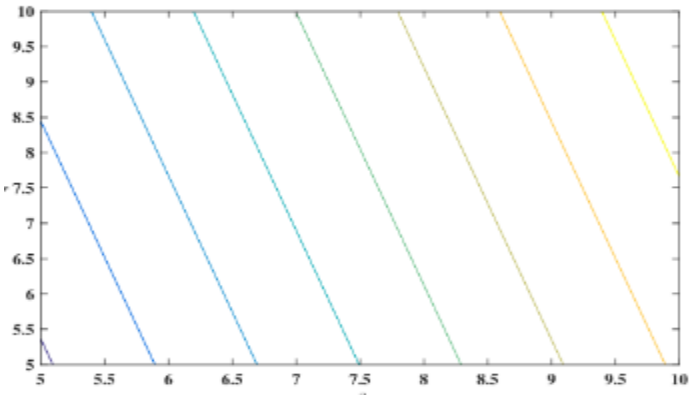


Figure 12: Contour plot of basic reproductive number $R0$ as a function of β and ξat

2 philadelphia hospitals in 2021," *JAMA*, vol. 329, no. 8, pp. 682–684, 2023.

[3] Z. Memon, S. Qureshi, and B. R. Memon, "Mathematical analysis for a new nonlinear measles epidemiological system using real incidence data from pakistan," *The European Physical Journal Plus*, vol. 135, no. 4, p. 378, 2020.

[4] M. A. Kuddus, M. Mohiuddin, and A. Rahman, "Mathematical analysis of a measles transmission dynamics model in bangladesh with double dose vaccination," *Scientific reports*, vol. 11, no. 1, p. 16571, 2021.

[5] O. J. Peter, H. S. Panigoro, M. A. Ibrahim, O. M. Otunuga, T. A. Ayoola, and A. O. Oladapo, "Analysis and dynamics of measles with control strategies: A mathematical modeling approach," *International Journal of Dynamics and Control*, vol. 11, no. 5, pp. 2538–2552, 2023.

[6] Z. Bai and D. Liu, "Modeling seasonal measles transmission in china," *Communications in Nonlinear Science and Numerical Simulation*, vol. 25, no. 1-3, pp. 19–26, 2015.

[7] Y. Xue, X. Ruan, and Y. Xiao, "Modelling the periodic outbreak of measles in mainland china," *Mathematical Problems in Engineering*, vol. 2020, no. 1, p. 3631923, 2020.

[8] J. Huang, S. Ruan, X. Wu, and X. Zhou, "Seasonal transmission dynamics of measles in china," *Theory in Biosciences*, vol. 137, no. 2, pp. 185–195, 2018.

[9] M. Roberts and M. Tobias, "Predicting and preventing measles epidemics in new zealand: application of a mathematical model," *Epidemiology & Infection*, vol. 124, no. 2, pp. 279–287, 2000.